Journal of Statistical Physics, Vol. 21, No. 2, 1979

On the Lifschitz Singularity and the Tailing in the Density of States for Random Lattice Systems

M. Romerio[1,2] **and W. Wreszinski**[3,4]

Received June 16, 1978; revised November 14, 1978

We prove rigorously the existence of a Lifschitz singularity in the density of states at zero energy in some random lattice systems of noninteracting bosons and fermions in any number ν of dimensions. The basic tool is a simple modification of the method of Fukushima to yield the correct upper and lower bounds for all ν. We also comment on the mathematical difference between the models treated and the system of phonons with mass disorder in the harmonic approximation, whose behavior is known to be of Debye form, not Lifschitz, at low temperatures.

KEY WORDS: Random lattice system; Lifschitz singularity; density of states; tailing; low-temperature behavior; random walk; Feynman–Kac formula.

1. INTRODUCTION

Several recent papers have been devoted to the study of the so-called Lifschitz singularity(11) of the average density of states $\langle g(\cdot)\rangle$ in random systems at zero energy.(7,12,16,18,19) This is an essential singularity of the type

$$\langle g(\omega)\rangle \underset{\omega\downarrow 0}{\sim} \exp[-\text{const}/\omega^{\nu/2}] \tag{1}$$

where ν is the number of space dimensions, and leads to a singularity of a similar type for some thermodynamic functions, such as the specific heat, at temperature zero. Lifschitz's derivation (Ref. 11; see also Ref. 16) is a beautiful but heuristic physical argument, which is also briefly discussed from the point of view of the connection with Griffiths' singularities(17) in Refs. 7

[1] Institute de Physique, Université de Neuchâtel, Neuchâtel, Switzerland.
[2] Supported by the Swiss National Science Foundation.
[3] Seminar für Theoretische Physik der ETH, Zürich, Switzerland.
[4] On leave of absence from the Instituto de Fisica, University of São Paulo, Brazil.

0022-4715/79/0800-0169$03.00/0

and 18. Lifschitz also gave arguments to evaluate the constant in the exponential in (1). We shall refer to the latter, for brevity, as the "Lifschitz constant" and to the exponent $\nu/2$ as the "Lifschitz exponent."

The problem has also been studied rigorously in some models.[1–4,8,9] In particular, the model with one-particle Hamiltonian $[-\Delta + q(a)]$ on $L^2(\mathbb{R})$, where $q(a)$, $a \in \mathbb{R}$, is a strong Markov process taking two values 0 and 1, has been treated in Ref. 3 (see also Ref. 4), and the same model on $L^2(\mathbb{R}^\nu)$, where $q(a)$, $a \in \mathbb{R}^\nu$, is a stationary Gaussian process, has been studied in Ref. 4 (although, however, neither the correct Lifschitz exponent nor the Lifschitz constant were obtained in the latter case). The one-dimensional model of a harmonic crystal with mass randomness introduced in Ref. 10 was treated rigorously from the present point of view in Ref. 8 and the general model of phonons with mass disorder in the harmonic approximation and any dimension was studied in Ref. 9. In Ref. 1 a powerful method, relying on some properties of the time-continuous random walk on $\mathbb{Z}^\nu$, was developed by Fukushima, which allowed the proof of upper and lower bounds of the form of the rhs of (1) in dimension $\nu = 1$ for a number of random lattice models. His proof was somewhat simplified in Ref. 2. In Section 3 we show that a simple modification of Fukushima's argument allows a proof of upper and lower bounds of the form of the rhs of (1) for all ν, for the same class of models treated in Ref. 1. To obtain (1) (or, alternatively, upper and lower bounds with the *same* constant in the exponential), we expect that the methods developed in Ref. 13 for the "Wiener sausage" should be more suitable.[5]

The quantity considered in Ref. 1 and here in Section 3, the average of the spectral measure of the Hamiltonian with respect to a certain vector, is equal (with probability one) to the "integrated density of states" of the (infinite) systems studied in Section 4 by a result due to Pasthur,[4] which is given without proof in Section 2, where we also introduce some notation. In Section 4 we prove that a model of bosons exhibits a singularity of type (1) and consequently a singularity of the specific heat at temperature $T = 0$ of type $\exp[-\mathrm{const}/T^{\nu/(\nu+2)}]$. For fermions, we exhibit a very simple model of decoupled bands, each of them corresponding to a one-particle "tight-binding" Hamiltonian of the type introduced in Section 2. As a corollary, we show that there exists a "tailing" of the average density of states in the gap having the Lifschitz form (1). We also comment on the mathematical difference between the models treated and the system of phonons with mass randomness in the harmonic approximation,[15] whose behavior is known to be of Debye, not Lifschitz, form at low temperatures.[9]

[5] In fact, a result analogous to the one of Ref. 13 for the lattice was obtained very recently by Donsker and Varadhan.[20] Although their result is more accurate (in particular, it provides the value of the Lifschitz constant), we preferred to keep the same elementary level of Fukushima's paper throughout, which is sufficient for our present purpose, i.e., exhibiting the singularity with the correct Lifschitz exponent.

2. MATHEMATICAL FRAMEWORK AND DENSITY OF STATES

Let $I \equiv \{\varepsilon_0 = 0, \varepsilon_1, \varepsilon_2, \ldots\}$, where $\{\varepsilon_i\}_{i=0,1,2,\ldots}$ is an increasing sequence (finite or infinite) of nonnegative real numbers, and associate a measure p_i to ε_i such that $\sum_{i\geqslant 0} p_i = 1$. Let $\Omega = \times_{a\in\mathbb{Z}^\nu} I_a$ where $\times_{a\in\mathbb{Z}^\nu}$ denotes the Cartesian product indexed by the points of $\mathbb{Z}^\nu$, together with the direct product (probability) measure, which we shall denote by P. Thus Ω becomes a compact topological space, with family $\mathscr{B}$ of Borel sets, and $(\Omega, \mathscr{B}, P)$ a probability space. Following mostly Ref. 1, we define a second-order difference operator H^0 on $\mathscr{H} = l^2(\mathbb{Z}^\nu)$ by

$$(H^0u)(a) = -\frac{\sigma^2}{2}\sum_{i=1}^{\nu}[u(a_1,\ldots, a_i - 1,\ldots, a_\nu) + u(a_1,\ldots, a_i + 1,\ldots, a_\nu) - 2u(\boldsymbol{a})], \qquad a\in\mathbb{Z}^\nu, \quad u\in C_0(\mathbb{Z}^\nu) \tag{2}$$

where σ is a positive constant, and $C_0(\mathbb{Z}^\nu)$ is the space of functions on $\mathbb{Z}^\nu$ with finite support. Let $\{q(a, \omega), a\in\mathbb{Z}^\nu, \omega\in\Omega\}$ be the family of "ath coordinate functions" of Ω, that is, random variables defined by

$$\omega \equiv (\omega_a)_{a\in\mathbb{Z}^\nu}: \qquad q(b, \omega) = \omega_b, \quad b\in\mathbb{Z}^\nu \tag{3}$$

The $q(a, \omega)$ are independent, identically distributed, nonnegative random variables defined on $(\Omega, \mathscr{B}, P)$. We define the operator

$$(H^\omega u)(a) = (H^0u)(a) + q(a, \omega)u(a), \qquad a\in\mathbb{Z}^\nu, \quad \omega\in\Omega \tag{4}$$

The operator H^ω considered as a linear operator defined on $C_0(\mathbb{Z}^\nu)$ is the sum of the bounded self-adjoint operator H^0 and the unbounded symmetric multiplication operator q, which is easily seen explicitly to be essentially self-adjoint on $C_0(\mathbb{Z}^\nu)$. Hence H^ω has a unique extension as a positive self-adjoint operator on $\mathscr{H}$, which we denote, as in Ref. 1, by $\bar{H}^\omega$. Express $\bar{H}^\omega = \int_{[0,\infty]} x\, dE_x{}^\omega$ by the associated spectral family $\{E_x{}^\omega, x\in\mathbb{R}\}$ and put $\rho^\omega(x) \equiv (E_xI_0, I_0)$, where $(\cdot\,,\,\cdot)$ denotes the l^2 inner product, and $I_a\ (a') = \delta_{a,a'}$, a, $a'\in\mathbb{Z}^\nu$. Denoting by $\langle\cdots\rangle$ expectation with respect to P, we set

$$\rho(x) \equiv \langle\rho^\cdot(x)\rangle \tag{5}$$

Let $\{\Lambda\}$ be a family of cubes centered at the origin in $\mathbb{Z}^\nu$, covering the whole of $\mathbb{Z}^\nu$, and P_Λ the operator of projection of $\mathscr{H}$ onto the subspace $C_0(\Lambda)$ of $C_0(\mathbb{Z}^\nu)$ consisting of all functions with support contained in Λ, and define

$$H_\Lambda{}^\omega \equiv P_\Lambda\bar{H}^\omega P_\Lambda \tag{6}$$

(This corresponds to "free boundary conditions" on Λ. It may be verified by standard arguments that Proposition 2.1 below is true independent of boundary conditions.) For each fixed Λ, $H_\Lambda{}^\omega$ has a discrete spectrum consisting of a finite number, $N(\Lambda)$ say, of eigenvalues $\{\varepsilon_i^{\Lambda,\omega}\}_{i=1}^{N(\Lambda)}$, and we may define the "integrated density of states" for the system Λ by

$$N_\Lambda{}^\omega(x) \equiv \frac{1}{|\Lambda|} \sum_{\varepsilon_i^{\Lambda,\omega} \leq x} 1; \qquad \omega \in \Omega, \quad x \in \mathbb{R}_+ \equiv [0, \infty) \tag{7}$$

where $|\Lambda|$ is the number of points in Λ. We remark that, in general, $N_\Lambda{}^\omega(\infty) \neq 1$, hence $N_\Lambda{}^\omega(\cdots)$ is a nonnormalized (right-continuous) distribution function.

Proposition 2.1.[4] For all $x \in \mathbb{R}_+$, $\lim_{|\Lambda| \to \infty} N_\Lambda{}^\omega(a, b] = \rho(a, b]$ almost everywhere with respect to P, at all continuity points $a \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$ of ρ, where ρ was defined in (5). (8)

The proof is the same as Pasthur's[4]: his condition that q be metrically transitive is trivially verified in the present case because the (ν-dimensional) shift operator in $\mathbb{Z}^\nu$ is ergodic (Ref. 14, p. 50; see also Ref. 8 for a similar application of this property to a related particular model).

Let $\dot{M} = (\dot{\Omega}, \dot{B}, \dot{X}_t, \dot{P}_a)$ be the (time-continuous) simple random walk on $\mathbb{Z}^\nu$,[1] where $\dot{P}_a$ is the probability measure governing sample paths $\dot{X}$ starting at a. Let $\dot{E}_a$ denote expectation with respect to $\dot{P}_a$. Then (Ref. 1; Ref. 22, p. 81) we have the Feynman–Kac formula:

$$(I_a, [\exp(-t\overline{H}^\omega)]I_a) = \dot{E}_a\left(\exp\left[-\int_0^t q(\dot{X}_s, \omega)\, ds\right]; \dot{X}_t = a\right) \tag{9}$$

This formula is a basic ingredient for the next section.

3. THE IMPROVED FUKUSHIMA BOUND

In this section we prove the following result:

Proposition 3.1

$$\overline{\lim_{x \downarrow 0}}[x^{\nu/2} \log \rho(x)] \leqslant -(\nu/2)^{\nu/2}(\nu + 2)^{-(\nu/2+1)}(\nu\sigma\beta_1/\sqrt{e})^{\nu/2} \tag{10}$$

where

$$\beta_1 \equiv |\log\langle \nu\sigma^2/[\nu\sigma^2 + 2q(0)]\rangle|$$

and

$$\underline{\lim_{x \downarrow 0}}[x^{\nu/2} \log \rho(x)] \geqslant -\left[\frac{\pi^2\sigma^2}{12|\log P(q(0) = 0)|}\right]^{\nu/2} \tag{11}$$

Proof. The upper bound follows precisely along the lines of Ref. 1: in particular, if

$$\dot{M}_t \geqslant \sup_{0 \leq s \leq t} |\dot{X}_s|$$

(in the notation of Section 2), one obtains, by the method of Ref. 1, the inequality

$$\overline{\lim_{t \to \infty}}\, t^{-\nu/(\nu+2)} \log \dot{E}_0(\exp -\beta \dot{M}_t) \leqslant -(\nu\beta\sigma/\sqrt{e})^{2/(\nu+2)} \tag{12}$$

Then (10) follows from (12) (Ref. 1, Lemma 4.3) and the exponential Tauberian theorem [Ref. 1, Theorem 2.1(ii)]. For the lower bound, let

$$k(t) \equiv \tilde{E}\left\{\exp\left[-\int_0^t q(\dot{X}_s)\, ds\right]; \dot{X}_t = 0\right\}$$

where $\tilde{E}$ denotes expectation with respect to the product measure $\tilde{P} = P \times \dot{P}_0$, and let $\dot{R}_t(\omega)$ denote the number of different points visited by the sample path $X_s(\omega)$ during the time interval $[0, t)$. Then, by Ref. 1, (26) (p. 82),

$$k(t) \geqslant \dot{E}_0(\exp -\beta_2 \dot{R}_t; \dot{X}_t = 0)$$

where $\beta_2 \equiv -\log P(q(0) = 0)$. Now, by an elementary geometric argument,

$$\dot{R}_t \leqslant (2\dot{M}_t + 3)^\nu$$

whence

$$k(t) \geqslant \dot{E}_0\{\exp[-\beta_2(2\dot{M}_t + 3)^\nu]; \dot{X}_t = 0\} \tag{13}$$

Let $G(x) \equiv \dot{P}_0(\dot{M}_t \leqslant x, \dot{X}_t = 0)$ for every real x. By Ref. 1, Lemma 3.1, and (13),

$$\begin{aligned}
k(t) &\geqslant 2\beta_2\nu \int_0^\infty dx\, \{\exp[-\beta_2(2x + 3)^\nu]\}(2x + 3)^{\nu-1} G(x) \\
&\geqslant 2^\nu(\sqrt{\nu})^\nu \beta_2\nu \int_0^\infty dx\, x^{\nu-1} \\
&\qquad \times \exp\{-[\beta_2(2x + 3)^\nu + \pi^2\sigma^2\nu^2 t/8x^2]\}(x + \sqrt{\nu})^{-\nu} \\
&= 2^\nu(\sqrt{\nu})^\nu \beta_2 \int_0^\infty dy\, (y^{1/\nu} + \sqrt{\nu})^{-\nu} \\
&\qquad \times \exp\{-[\beta_2(2y^{1/\nu} + 3)^\nu + \pi^2\sigma^2\nu t/8y^{2/\nu}]\} \equiv F(t)
\end{aligned}$$

Let $\varepsilon > 0$ and $C > 0$ be for the moment arbitrary constants, to be chosen later. Then

$$\begin{aligned}F(t) &\geqslant 2^{\nu}(\sqrt{\nu})^{\nu}\beta_2 \int_0^{Ct^{\varepsilon}} dy\,(y^{1/\nu} + \sqrt{\nu})^{-\nu} \exp\{-[\beta_2(2y^{1/\nu} + 3)^{\nu} \\ &\quad + \pi^2\sigma^2\nu t/8y^{2/\nu}]\} \\ &\geqslant 2^{\nu}(\sqrt{\nu})^{\nu}\beta_2(Ct^{\varepsilon/\nu} + \sqrt{\nu})^{-\nu} \\ &\quad \times \exp[-\beta_2(2Ct^{\varepsilon/\nu} + 3)^{\nu}] \int_0^{Ct^{\varepsilon}} \exp(-\pi^2\nu\sigma^2 t/8y^{2/\nu})\,dy \\ &\geqslant Ct^{\varepsilon}2^{\nu}(\sqrt{\nu})^{\nu}\beta_2(Ct^{\varepsilon/\nu} + \sqrt{\nu})^{-\nu}\exp[-\beta_2(3Ct^{\varepsilon/\nu})^{\nu}] \\ &\quad \times \exp[-(\pi^2\nu\sigma^2/8C^{2/\nu})t^{1-2\varepsilon/\nu}]\end{aligned}$$

for t sufficiently large. Choosing now

$$\varepsilon = \nu/(\nu + 2)$$

it follows that $1 - 2(\varepsilon/\nu) = \nu/(\nu + 2)$. Hence

$$\varliminf_{t\to\infty} t^{-\nu/(\nu+2)} \log \kappa(t) \geqslant -3\beta C - (\pi^2\sigma^2\nu/8C^{2/\nu}) \tag{14}$$

for all $C > 0$. Taking now the maximum (in $C > 0$) of the rhs of (14), we find

$$\varliminf_{t\to\infty} t^{-\nu/(\nu+2)} \log \kappa(t) \geqslant -(\pi^2\sigma^2/12\beta_2)^{\nu/(\nu+2)} \tag{15}$$

Relation (11) follows directly from (15) and the exponential Tauberian theorem [Ref. 1, Theorem 2.1(i)]. ■

Remark 3.1. Under the conditions

$$0 < P(q(0) = 0) < 1 \tag{16}$$

it follows from (10) and (11) that there exist constants

$$0 < A < \infty, \qquad 0 < B < \infty \tag{17}$$

such that

$$-A \leqslant \varliminf_{x\downarrow 0} [x^{\nu/2} \log \rho(x)] \leqslant \varlimsup_{x\downarrow 0} [x^{\nu/2} \log \rho(x)] \leqslant -B \tag{18}$$

Both conditions (16) are quite natural; the first states that energy zero gives a nonzero contribution to the spectrum and the density of states, and the second excludes a perfect (i.e., nonrandom) system.

4. MODELS

As mentioned in the introduction, the models of random lattice systems studied rigorously so far in the literature from the point of view of low-temperature behavior and the Lifschitz singularity belong to two different classes:

(a) Noninteracting bosons or fermions with random impurities, described by Hamiltonians of type (4);
(b) The harmonic crystal with random masses (see, e.g., Ref. 9 for the Hamiltonian), with a probability mass distribution with finite greatest mass and strictly positive lowest mass.

Concerning low-energy (or low-temperature) behavior, there is a profound difference between the above classes: as pointed out in Ref. 9, low-temperature behavior of the specific heat of systems of class (b) follows the Debye law, although a Lifschitz-type singularity was rigorously proven to exist[8] (see also Ref. 7) for a limiting case of (b), namely the one-dimensional model of Ref. 10, where the mass at each site is an independent random variable taking two values, m and M, with probabilities p and $1 - p$, and M is infinite. As remarked in Ref. 9, one might expect that a Lifschitz-type behavior of the specific heat of the form $\exp[-\text{const}/T^{\nu/(\nu+2)}]$, as we shall see (see also Ref. 7), would manifest itself as an in-band resonance at higher temperatures.

In contrast, the low-temperature behavior of models of class (a) is of the Lifschitz form, as we shall see presently. [Of course, the behavior of class (b) models is physically expected in view of the interpretation of the coefficient of the linear term in the average density of states as the velocity of sound in the medium.] For clarity, we show now where the proof of Ref. 9 fails for model (4). Suppose for simplicity that $q(a, \cdot)$ takes for each $a \in \mathbb{Z}^\nu$ only the values $\varepsilon_0 = 0$ and $\varepsilon > 0$, and denote by H^0 and H^ε (resp. $H_\Lambda{}^0$ and $H_\Lambda{}^\varepsilon$) the Hamiltonians (4) [resp. (6)] where q is replaced by the *constants* $\varepsilon_0 = 0$ and ε, respectively. Then, for all $\omega \in \Omega$,

$$H_\Lambda{}^0 \leqslant H_\Lambda{}^\omega \leqslant H_\Lambda{}^\varepsilon$$

This yields, for the integrated density of states [defined as in (7)]:

$$N_\Lambda{}^\varepsilon(x) \leqslant N_\Lambda{}^\omega(x) \leqslant N_\Lambda{}^0(x), \qquad \forall \omega \in \Omega, \quad \forall x \in \mathbb{R}_+$$

and hence, for the limits as $|\Lambda| \to \infty$, with probability one,

$$\rho^\varepsilon(x) \leqslant \rho(x) \leqslant \rho^0(x) \tag{19}$$

It follows, however, that $\rho^\varepsilon(x) = 0$ for $0 \leqslant x < \varepsilon$; therefore, (19) yields an inequality in only one direction for the behavior as $x \downarrow 0$ of $\rho(x)$ (namely, an upper bound). In the case of model (b), however, the quantities corresponding to ρ^0 and ρ^ε in (19) refer to the greatest and smallest mass, respectively, and their behavior as $x \downarrow 0$ is qualitatively the same.

We consider now some possible models of class (a), to which the proofs in Sections 2 and 3 apply.

4.1. Boson Systems

One may take (4) to be the one-boson Hamiltonian of a system of noninteracting spin waves (i.e., Heisenberg Hamiltonian "in the one-spin-wave approximation"), with random impurities. For each $\omega \in \Omega$ and $\beta \in (0, \infty)$, let $C^{\omega}_{\Lambda,\beta}$ denote the specific heat of this (random) system enclosed in a cubic box Λ as in Section 2, and define the function

$$x \in \mathbb{R}_+ \to h_\beta(x) \equiv \left[\frac{\beta x/2}{\sinh(\beta x/2)}\right]^2 \tag{20}$$

By a standard result (see, e.g., Ref. 15, p. 46), and in the notation of Section 2,

$$\begin{aligned} C^{\omega}_{\Lambda,\beta} &= \frac{1}{|\Lambda|}\operatorname{tr}_\Lambda[h_\beta(H_\Lambda{}^\omega)] = \frac{1}{|\Lambda|}\sum_{i=1}^{N(\Lambda)} h_\beta(\varepsilon_i^{\Lambda,\omega}) \\ &= \int_0^\infty h_\beta(x)\, dN_\Lambda{}^\omega(x) \qquad (\operatorname{tr}_\Lambda = \operatorname{tr}_{l^2(\Lambda)}) \end{aligned} \tag{21}$$

Lemma 4.1. Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a bounded, continuous function, and let

$$f(\Lambda, \omega) \equiv \int_0^\infty f(x)\, dN_\Lambda{}^\omega(x) \tag{22}$$

Then

$$\lim_{|\Lambda|\to\infty} f(\Lambda, \omega) = \int_0^\infty f(x)\, d\rho(x) \qquad \text{a.e. with respect to } P$$

Proof. The result follows from Proposition 2.1 and a standard theorem on the vague convergence of measures (see, e.g., Theorem 4.5.4, p. 198, of Ref. 5). ∎

Corollary 4.1. For each $\beta \in (0, \infty)$,

$$C_\beta \equiv \lim_{|\Lambda|\to\infty} C^{\omega}_{\Lambda,\beta} = \int_0^\infty h_\beta(x)\, d\rho(x) \tag{23}$$

a.e. with respect to P.

Corollary 4.2. There exists a $\delta > 0$ and strictly positive finite constants C_1, C_2, λ_1, and λ_2 such that, for all $0 \leqslant T \leqslant \delta$,

$$C_2 \exp(-\lambda_2/T^{\nu/(\nu+2)}) \leqslant C_\beta \leqslant C_1 \exp(-\lambda_1/T^{\nu/(\nu+2)}) \tag{24}$$

Proof. This is a sort of "Abelian" theorem in the temperature and may be proven, for instance, by splitting the interval $[0, \infty)$ in (23) in the following way:

$$[0, \infty) = [0, T^{2/(\nu+2)}) \bigcup [T^{2/(\nu+2)}, \infty)$$

One then applies (10) and (11) to each part, as in Ref. 7.[6] ∎

4.2. Fermion Systems

We consider here a very simple and idealized model of two decoupled bands. The perfect system is described by the Hamiltonian [on $\mathscr{H} \otimes \mathscr{H}$, where $\mathscr{H} = l^2(\mathbb{Z}^\nu)$]

$$H = (-H^0 - V_1/2) \otimes \mathbb{1} + \mathbb{1} \otimes (H^0 + V_1/2)$$

where V_1 is a positive constant (the "band gap"), and H^0 is the "tight-binding" Hamiltonian (3). The spectrum Σ_H of H is given by (Ref. 11, Theorem VIII-33, Corollary)

$$\Sigma_H = [-\Sigma_{H^0} - V_1/2, -V_1/2] \bigcup [V_1/2, \Sigma_{H^0} + V_1/2]$$

where Σ_{H^0}, the spectrum of H^0, is an absolutely continuous "band" starting at zero, as is easily proved from the explicit form (2). The random system is described by the Hamiltonian

$$\tilde{H}^\omega \equiv \bar{H}_1{}^\omega \otimes \mathbb{1} + \mathbb{1} \otimes \bar{H}_2{}^\omega, \qquad \omega \in \Omega$$

on $\mathscr{H} \otimes \mathscr{H}$, where

$$\bar{H}_1{}^\omega \equiv \bar{H}^\omega + V_1/2, \qquad \bar{H}_2{}^\omega \equiv -\bar{H}^\omega - V_1/2$$

and $\bar{H}^\omega$ is the same operator defined in Section 2 [from (4)], where, however, the random variables $q(a, \omega)$ are assumed to take, for each $a \in \mathbb{Z}^\nu$, the possible values in the set

$$I \equiv \{\varepsilon_0 = -V_2, \varepsilon_1, \ldots, \varepsilon_N = +V_2\}$$

where

$$0 < V_2 < V_1/2$$

[6] Some obvious modifications should be made in this proof to allow for the fact that the support of $\rho(\cdot)$ may be pure point, everywhere dense on a subset of $\mathbf{R}_+$ containing the origin, as is the case in the model introduced in Ref. 10 (see Ref. 8), as well as in all models treated in the present paper, at least when $\nu = 1$.[19]

is the "spreading constant." Hence, the spectrum Σ^{ω} of $\tilde{H}^{\omega}$ is given by (Ref. 21, Theorem VIII-33, Corollary)

$$\Sigma^{\omega} = \Sigma(\bar{H}_1{}^{\omega}) + \Sigma(\bar{H}_2{}^{\omega})$$

and is therefore contained in the union of the sets

$$[-\Sigma_{H^0} - V_1/2 - V_2, -(V_1/2 - V_2)]$$

and

$$[V_1/2 - V_2, \Sigma_{H^0} + V_1/2 + V_2]$$

By a proof analogous to that in Section 2, the "integrated density of states" $\tilde{\rho}$ of the random system (defined as in Section 2) is, with probability one,

$$\tilde{\rho}(x) = \rho_1(x) + \rho_2(x), \qquad x \in \mathbb{R} \tag{25a}$$

where

$$\rho_1(x) \equiv \rho(x - (V_1/2 - V_2)) \tag{25b}$$

and

$$\rho_2(x) \equiv \rho(-x + (V_1/2 - V_2)) \tag{25c}$$

where ρ is the *same* quantity defined in Section 2. We see from (25) that $\tilde{\rho}$ presents in the regions $[V_2, V_1/2]$ and $[-V_1/2 - V_2]$ a "tailing" relative to the density of states of the perfect system (which has support contained in Σ_H), which is of the Lifschitz form near the end points $(V_1/2 - V_2)$ and $-(V_1/2 - V_2)$.

Clearly, this model is *constructed* so as to exhibit the Lifschitz "tailing" phenomenon, which has received so much attention in the theory of random media (see Ref. 22 for a nice review). We hope, however, to obtain a behavior of the type described in this section from a more realistic model, inspired by Ref. 23, where the physical meaning of the constants V_1 and V_2 will also be better elucidated. This will be dealt with elsewhere.[24]

ACKNOWLEDGMENTS

One of us (WFW) should like to thank Prof. S. R. S. Varadhan for communicating to him the results of Ref. 20 in advance of publication, and the referee for a number of useful suggestions and remarks.

REFERENCES

1. M. Fukushima, *Osaka J. Math.* **11**:73 (1974).
2. H. Nagai, *Osaka J. Math.* **14**:111 (1977).
3. M. Benderskii and L. A. Pasthur, *Math. Sb.* **82**:273 (1970).

4. L. A. Pasthur, *Russ. Math. Surveys* **28**:1 (1973).
5. R. B. Ash, *Real Analysis and Probability* (Academic Press, 1972).
6. T. Kato, *Perturbation Theory for Linear Operators* (Springer Verlag, 1966).
7. W. F. Wreszinski, *Phys. Lett.* **62A**:15 (1977).
8. L. van Hemmen, *Phys. Lett.* **64A**:177 (1977).
9. J. Hammerberg, *Phys. Lett.* **65A**:385 (1978).
10. C. Domb, A. A. Maradudin, E. W. Montroll, and G. H. Weiss, *Phys. Rev.* **115**:24 (1959).
11. I. M. Lifschitz, *Sov. Phys.—Usp.* **7**:549 (1965); *Adv. Phys.* **13**:483 (1964).
12. J. M. Luttinger, *Phys. Rev. Lett.* **37**:609 (1976).
13. M. Donsker and S. R. S. Varadhan, *Comm. Pure Appl. Math.* **28**:525 (1975).
14. P. Walters, *Ergodic Theory, Introductory Lectures* (Springer Verlag, 1975).
15. A. A. Maradudin, E. W. Montroll, and G. H. Weiss, *Solid State Phys., Suppl. 3* (1963).
16. R. Friedman and J. M. Luttinger, *Phys. Rev. B* **12**:4460 (1975).
17. R. B. Griffiths, *Phys. Rev. Lett.* **23**:17 (1969).
18. A. B. Harris, *Phys. Rev. B* **12**:203 (1975).
19. I. Goldsh'tein, S. Molchanov, and L. Pasthur, *Funct. Anal. Appl.* **11**:1 (1977).
20. M. Donsker and S. R. S. Varadhan, Courant Inst. preprint (1978).
21. M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I—Functional Analysis* (Academic Press, 1972).
22. M. Lax, Wave Propagation and Conductivity in Random Media, in *Stochastic Differential Equations* (SIAM–AMS Proceedings), J. B. Keller and H. P. McKean, eds. (AMS, Providence, Rhode Island, 1973).
23. G. G. Hall, *Phil. Mag.* **43**:338 (1952); see also D. Weaire, *Phys. Rev. Lett.* **26**:1541 (1971).
24. M. Romerio and W. F. Wreszinski, in preparation.